

# ELEMENTARY SUBMODELS AND SEPARABLE MONOTONICALLY NORMAL COMPACTA

TODD EISWORTH

**ABSTRACT.** In this note, we use elementary submodels to prove that a separable monotonically normal compactum can be mapped onto a separable metric space via a continuous function whose fibers have cardinality at most 2.

## 1. INTRODUCTION

Mary Ellen Rudin's 2001 proof of Nikiel's Conjecture [15] is a tour de force of combinatorial set theory and general topology. The work in this note arose from the author's attempt at understanding her proof; it became clear upon reading her paper that elementary submodels ought to be able to shed light on the structure of monotonically normal compacta. In the sequel, we will give one such application.

We will begin with the definition of monotonically normal spaces.

**Definition 1.1.** A space<sup>1</sup>  $X$  is monotonically normal if there is a binary function  $H$  whose domain consists of all pairs  $(p, U)$  where  $p \in U$  and  $U$  is open in  $X$  such that

- (1)  $H(p, U)$  is an open set such that  $p \in H(p, U) \subseteq U$ ,
- (2) if  $p \notin V$  and  $q \notin U$  then  $H(p, U) \cap H(q, V) = \emptyset$ , and
- (3) if  $V \subseteq U$  is another open set, then  $H(p, U) \subseteq H(p, V)$ .

The function  $H$  is called a *monotone normality operator* on  $X$ .

It is easy to see that a monotonically normal space  $X$  is normal — if  $H$  and  $K$  are disjoint closed subsets of  $X$ , and we define

$$(1.1) \quad U = \bigcup_{p \in H} H(p, X \setminus K)$$

and

$$(1.2) \quad V = \bigcup_{q \in K} H(q, X \setminus H),$$

---

*Date:* February 2, 2008.

1991 *Mathematics Subject Classification.* Primary: 54E20; Secondary: 54F05.

*Key words and phrases.* compact space, elementary submodels, monotonically normal, metric space.

The author acknowledges support from NSF grant DMS-0506063.

<sup>1</sup>We assume that all topological spaces under consideration in the paper are Tychonoff.

then  $U$  and  $V$  provide the required separation of  $H$  and  $K$ .

There is an extensive literature devoted to monotonically normal spaces (Gruenhage's two articles [10] and [11] can be consulted for more information); much of Rudin's contribution to this literature centered on Nikiel's Conjecture, and her work culminated in the following theorem.

**Theorem 1** (Nikiel's Conjecture – Rudin [15]). *A compact Hausdorff space  $X$  is monotonically normal if and only if it is the continuous image of a linearly ordered compactum.*

Our work in this area begins with her proof, but before we can prove our theorem, we must deal with some aspects of elementary submodels in topology.

## 2. ELEMENTARY SUBMODELS AND $X/M$

We will assume that the reader has some familiarity with the use of elementary submodels in topology — Dow's papers [4] and [6] provide more than adequate preparation. We are going to be using a particular construction involving elementary submodels due independently to Bandlow ([3], [2], and [1]) and Dow [5], and this section of the paper will be used to develop the theory in a self-contained manner.

Let us assume now that  $(X, \tau)$  is a Tychonoff space,  $\chi$  is a “sufficiently large” regular cardinal, and  $M$  is an elementary submodel of  $H(\chi)$ , the collection of sets hereditarily of cardinality  $< \chi$ .

**Definition 2.1.** Two points  $x$  and  $y$  are  $M$ -equivalent, denoted  $x =_M y$ , if  $f(x) = f(y)$  for every continuous function  $f : X \rightarrow \mathbb{R}$  such that  $f \in M$ .

**Proposition 2.2.**  $M$ -equivalence is an equivalence relation, and the equivalence classes are closed subsets of  $X$ .

*Proof.* The fact that  $=_M$  is an equivalence relation is trivial. To see that equivalence classes are closed, suppose that  $x$  is not  $M$ -equivalent to  $y$ , and let  $f \in M$  be a function that witnesses this. Then  $f^{-1}(\mathbb{R} \setminus \{f(y)\})$  is an open neighborhood of  $x$  that is disjoint to the  $M$ -equivalence class of  $y$ .  $\square$

We denote the equivalence class of  $x$  by  $[x]$ , and let  $X/M$  denote the set of all  $M$ -equivalence classes of points in  $X$ . There are several natural choices for topologizing  $X/M$  (all of which coincide if  $X$  is compact), but the theory works smoothest when the following definition is used.

**Definition 2.3.** Let  $\pi_M$  be the natural projection of  $X$  onto  $X/M$ . We topology  $X/M$  by taking as a base all sets of the form  $\pi_M[U]$  for  $U$  a co-zero set in  $M$ .

The proof of the following proposition is left to the reader — all of the clauses enumerated follow easily from the definitions involved.

**Proposition 2.4.** Let  $X$  and  $M$  be as above.

- (1) The mapping  $\pi_M : X \rightarrow X/M$  is continuous.
  - (2)  $x =_M y$  if and only if for every co-zero  $U \in M$ ,
- (2.1) 
$$x \in U \iff y \in U.$$
- (3)  $[x] = \bigcap\{Z \in M : x \in Z, Z \text{ a zero-set}\}.$
  - (4)  $[x] = \bigcap\{\overline{U} : x \in U \in M, U \text{ a co-zero set}\}.$
  - (5)  $X/M$  is a Hausdorff space.

We will prove shortly that  $X/M$  is in fact a Tychonoff space, but before we do that we point out the connection between  $X/M$  and a more typical construction using elementary submodels.

**Definition 2.5.** Given  $X$  and  $M$  as above,  $X_M$  is defined to be the topological space whose underlying set is  $M \cap X$  with the topology generated by sets of the form  $M \cap U$ , where  $U$  is an open subset of  $X$  that is in  $M$ .

**Proposition 2.6.**  $X_M$  is homeomorphic to a dense subspace of  $X/M$ .

*Proof.* The proof is the obvious one — a point  $x \in M \cap X$  is sent to its equivalence class in  $X/M$ . The topologies involved are defined in such a way that this map is a homeomorphism of  $X_M$  onto its image, and the fact that this image is dense in  $X/M$  follows easily as any non-empty open subset of  $X$  that is in  $M$  must contain an element of  $M \cap X$ .  $\square$

Our next goal is to give a more concrete description of  $X/M$  in terms of  $\beta X$ , the Stone-Cech compactification of  $X$ . One of the standard constructions of  $\beta X$  involves embedding  $X$  into a product  $[0, 1]^\kappa$  of unit intervals, so we first investigate the nature of  $X/M$  for  $X$  of the form  $[0, 1]^\kappa$ .

**Proposition 2.7.** Suppose  $X = [0, 1]^\kappa$  for some cardinal  $\kappa$  (so we view points of  $X$  as functions from  $\kappa$  to  $[0, 1]$ ), and let  $M$  be an elementary submodel of  $H(\chi)$  containing  $X$ . Then

- (1)  $x =_M y$  if and only if  $x \upharpoonright M \cap \kappa = y \upharpoonright M \cap \kappa$ , and
- (2)  $X/M$  is homeomorphic to  $[0, 1]^{M \cap \kappa}$ .

*Proof.* It is certainly the case that  $x \upharpoonright M \cap \kappa = y \upharpoonright M \cap \kappa$  if  $x =_M y$ , as projection onto the  $\alpha$ th component is a real-valued function on  $X$  that is in  $M$  if  $\alpha \in M \cap \kappa$ . Suppose now that  $x \neq_M y$ ; we produce an  $\alpha \in M \cap \kappa$  such that  $x(\alpha) \neq y(\alpha)$ . The key to this is a well-known fact that a continuous real-valued function defined on a product of compact spaces depends on countably many coordinates.<sup>2</sup> In our context, this means if we have a function  $f : X \rightarrow \mathbb{R}$  then there is a countable set  $S \subseteq \kappa$  such that  $f(x) = f(y)$  if  $x \upharpoonright S = y \upharpoonright S$ . If  $x \neq_M y$ , then there is a function  $f \in M$  mapping  $X$  to  $\mathbb{R}$  such that  $f(x) \neq f(y)$ . The model  $M$  knows that  $f$  depends on countably many coordinates, so there is a countable  $S \in M$  with the requisite properties. In particular, there is an  $\alpha \in S$  such that

---

<sup>2</sup>See Exercise 3.2H(a) of Engelking [8], for example.

$x(\alpha) \neq y(\alpha)$ . Since the set  $S$  is countable and an element of  $M$ , we know that  $S \subseteq M$  and therefore there is an  $\alpha \in S \subseteq M \cap \kappa$  for which  $x(\alpha) \neq y(\alpha)$ , as required.

Thus, there is a natural correspondence between  $X/M$  and  $[0, 1]^{M \cap \kappa}$ . The fact that this correspondence is a homeomorphism is not difficult to show, and is left to the reader.  $\square$

Now suppose we have  $X$  and  $M$  as usual. Since  $\chi$  is “large enough”, we know that the space  $C^*(X)$  of continuous functions from  $X$  to  $[0, 1]$  is going to be an element of  $M$ , as it is definable from  $X$  using parameters available in  $M$ . Thus, inside  $M$  there will be an enumeration  $\langle f_\alpha : \alpha < \kappa \rangle$  of  $C^*(X)$ .<sup>3</sup>

We know that the function  $e : X \hookrightarrow [0, 1]^\kappa$  that maps  $x$  to  $\langle f_\alpha(x) : \alpha < \kappa \rangle$  embeds  $X$  as a subspace of  $[0, 1]^\kappa$ . The following proposition establishes a similar connection between  $X/M$  and  $[0, 1]^{M \cap \kappa}$ .

**Proposition 2.8.** Suppose  $X$ ,  $M$ , and  $\langle f_\alpha : \alpha < \kappa \rangle$  are as in the preceding discussion. Then there is a natural embedding  $e/M$  making the following diagram commute:

$$(2.2) \quad \begin{array}{ccc} X & \xrightarrow{e} & [0, 1]^\kappa \\ \pi/M \downarrow & & \downarrow \pi/M \\ X/M & \xrightarrow[e/M]{} & [0, 1]^{M \cap \kappa} \end{array}$$

The preceding material is taken from Section 5 of Dow’s [5]; the reader can find a more detailed discussion there. For our purposes, we need only the following corollary.

**Corollary 2.9.**  $X/M$  is a Tychonoff space.

*Proof.* The embedding  $e/M$  from the preceding proposition shows us that  $X/M$  is homeomorphic to a subspace of the compact space  $[0, 1]^{M \cap \kappa}$ .  $\square$

### 3. MONOTONE NORMALITY

We are now in a position to state the main theorem of this note.

**Theorem 2.** *Let  $X$  be a separable monotonically normal compactum, and let  $M$  be a countable elementary submodel of  $H(\chi)$  containing  $X$ . Then each  $=_M$ -equivalence class has cardinality at most 2.*

As a corollary, we get the following result mentioned in the abstract.

---

<sup>3</sup>There will be many such enumerations and it doesn’t really matter which one we use, but for definitiveness, we can consider  $M$  to be an elementary submodel of the expanded structure  $\langle H(\chi), \in, <_\chi \rangle$  where  $<_\chi$  is a fixed well-ordering of  $H(\chi)$ . We can then use the well-ordering to pick the particular enumeration we use.

**Corollary 3.1.** If  $X$  is a separable monotonically normal compactum, then  $X$  can be continuously mapped onto a separable metric space by a 2-to-1 function<sup>4</sup>.

*Proof.* In light of Theorem 2, it suffices to note that  $X/M$  is compact (as it is the continuous image of  $X$ ) and of countable weight (as the countably many co-zero sets in  $M$  define a base).  $\square$

Actually, much more can be shown. In particular,  $X$  can be written as the inverse limit of a system of compact metric spaces in such a way that all of the projections from  $X$  onto members of the system are two-to-one functions.

Before we prove Theorem 2, we deal with the special case where  $X$  is linearly ordered as this is particularly simple, and it hints at why Theorem 2 is connected to Nikiel's Conjecture.

Suppose now that  $X$  is a separable linearly ordered space, and let  $M$  be a countable elementary submodel of  $H(\chi)$  containing  $X$ . In the model  $M$ , one can find a countable set  $D \subseteq X$  that is dense in  $X$ . Since  $D$  is countable, it follows that every element of  $D$  is also in  $M$ .

Now suppose  $x < y$  are  $M$ -equivalent; it suffices to prove that the interval  $(x, y)$  is empty. Assume by way of contradiction that  $(x, y) \neq \emptyset$ . Then there is an element  $d \in D$  such that  $x < d < y$ . If one of  $(x, d)$  and  $(d, y)$  is empty, then we easily get a function in  $M$  separating  $x$  and  $y$  — for example, if  $(x, d)$  is empty then  $x$  is in  $M$  (it's definable in  $M$  as the predecessor of  $d$ ), and hence so is the (continuous) function sending  $(-\infty, x]$  to 0 and  $[d, \infty)$  to 1. If both  $(x, d)$  and  $(d, y)$  are non-empty, then we can find  $d_1$  and  $d_2$  in  $D$  such that

$$(3.1) \quad x < d_1 < d_2 < y$$

and the disjoint closed sets  $(-\infty, d_1]$  and  $[d_2, \infty)$  are in  $M$  and separate  $x$  and  $y$ . Since  $X$  is normal, we can find in  $M$  a function separating these two closed sets, and the function also separates  $x$  and  $y$ .

Thus, if  $X$  is separable and linearly ordered (we don't need compactness for this case) and  $M$  is as above, then each  $=_M$  equivalence class is either a singleton, or a pair  $x < y$  with  $y$  the immediate successor of  $x$ .

*Proof of Theorem 2.* Let  $X$  be a separable monotonically normal compactum, and let  $M$  be a countable elementary submodel of  $H(\chi)$  that contains  $X$ . Let  $H \in M$  be a monotone normality operator.

**Lemma 3.2.** If  $K$  is an  $=_M$ -equivalence class and  $U$  is any open neighborhood of  $K$ , then there is a co-zero set  $V \in M$  such that  $K \subseteq V$  and  $\overline{V} \subseteq U$ .

---

<sup>4</sup>We operate under the convention that this means pre-images of points have cardinality at most 2.

*Proof.* We know that  $K$  is the intersection of all co-zero sets in  $M$  that contain it. Since  $X$  is compact and  $K$  is closed, this implies that the collection of co-zero sets from  $M$  that contain  $K$  is a base for  $K$ , i.e., any open set containing  $K$  must contain such a co-zero set. The result follows immediately.  $\square$

**Lemma 3.3.** If  $K$  is an  $=_M$ -equivalence class and  $U$  is any open neighborhood of  $K$ , then there is a point  $p \in M \cap U$  and a co-zero set  $V \in M$  such that

- (1)  $p \in M \cap V$ ,
- (2)  $\overline{V} \subseteq U$ , and
- (3)  $K \subseteq H(p, V)$ .

*Proof.* As in the proof of Lemma 3.2, there is a co-zero set  $V \in M$  such that  $K \subseteq V$  and  $\overline{V} \subseteq U$ . Since  $V$  is (in  $M$ ) a countable union of zero sets, there is a zero set  $Z \in M$  such that  $Z \cap K \neq \emptyset$  and therefore  $K \subseteq Z$  by the definition of  $M$ -equivalence. For each  $p \in Z$  there is a co-zero set  $V_p$  such that

$$(3.2) \quad p \in V_p \subseteq H(p, V).$$

By elementarity, we may assume that the mapping  $p \mapsto V_p$  is an element of  $M$ , and thus  $\{V_p : p \in Z\}$  is an open cover of  $Z$  that is an element of  $M$ . Since  $Z$  is compact, there is finite  $Z_0 \subseteq Z$  such that

$$(3.3) \quad Z \subseteq \bigcup_{z \in Z_0} V_p.$$

We may assume that  $Z_0 \in M$ , and therefore  $Z_0 \subseteq M$ . Thus, there is a  $p \in M \cap Z$  such that  $V_p \cap K \neq \emptyset$ . Since  $V_p$  is a co-zero set in  $M$ , it follows that  $K \subseteq V_p$ . Thus

$$(3.4) \quad K \subseteq V_p \subseteq H(p, V) \subseteq V,$$

and the result follows.  $\square$

We now are in a position to apply the monotone normality of  $X$  in a non-trivial way.

**Proposition 3.4.** Suppose  $K$  is a  $=_M$ -equivalence class,  $\{x_0, x_1\} \subseteq K$ , and  $W_0$  and  $W_1$  are disjoint open sets with  $x_i \in W_i$ .<sup>5</sup> If  $K'$  is any other equivalence class, then there is at most one  $i < 2$  with  $K \cap H(x_i, W_i) \neq \emptyset$ .

*Proof.* Let  $U$  and  $U'$  be disjoint co-zero sets in  $M$  separating  $K$  and  $K'$ . By the previous lemma, there is a point  $p \in M \cap U'$  such that

$$(3.5) \quad K' \subseteq H(p, U').$$

---

<sup>5</sup>So  $W_0$  and  $W_1$  won't be elements of  $M$ .

Note that  $p$  is an element of  $W_i$  for at most one  $i$ , and that  $K \cap U' = \emptyset$ . If  $p \notin W_i$ , then

$$(3.6) \quad H(p, U') \cap H(x_i, W_i) = \emptyset$$

because  $H$  is a monotone normality operator. Since  $K' \subseteq H(p, U')$ , it follows that

$$(3.7) \quad K' \cap H(x_i, W_i) = \emptyset$$

as well.  $\square$

**Definition 3.5.** Let  $U \in M$  be a co-zero set. An equivalence class  $K \in X/M$  is said to be *shattered by  $U$*  if  $K \subseteq U$  and there exist  $\{x_i : i < 3\}$  and  $\{W_i : i < 3\}$  such that

- $\{x_i : i < 3\} \subseteq K$ ,
- $W_i$  is an open neighborhood of  $x_i$ ,
- the  $W_i$ 's have pairwise disjoint closures, and
- $H^4(x_i, W_i) \not\subseteq U$  for all  $i < 3$ .

(Here the notation  $H^n(p, U)$  is defined by induction:  $H^2(p, U) = H(H(p, U))$  and  $H^{n+1}(p, U) = H(H^n(p, U))$ .)

We will show that each co-zero  $U \in M$  can shatter at most countably many elements of  $X/M$ , and that every  $K \in X/M$  of cardinality greater than 2 is shattered by some  $U \in M$ .

**Proposition 3.6.** Let  $U \in M$  be a co-zero set. Then there are at most countably many equivalence classes in  $X/M$  that are shattered by  $U$ .

*Proof.* By way of contradiction, suppose there are uncountably many equivalence classes shattered by  $U$ . Since  $U$  is a co-zero set in  $M$ , we know there is a family  $\{Z_n : n \in \omega\}$  of zero-sets in  $M$  whose union is  $U$ . Recall that since  $Z_n \in M$ , if  $Z_n$  meets an equivalence class in  $X/M$ , then  $Z_n$  actually contains the entire equivalence class. Thus we can find a zero-set  $Z \in M$  and a family  $\{K_n : n \in \omega\}$  of equivalence classes in  $X/M$  such that

- $Z \subseteq U$
- $K_n \subseteq Z$  for all  $n$
- each  $K_n$  is shattered by  $U$

For  $n < \omega$ , let  $\{x_i^n : i < 3\}$  and  $\{W_i^n : i < 3\}$  be as in the previous definition for the equivalence class  $K_n$ .

For each pair of natural numbers  $m < n$ , by Proposition 3.4 we can find a value  $i = i(m, n) < 3$  such that

$$(3.8) \quad K_n \cap H(x_i^m, W_i^m) = \emptyset \text{ and } K_m \cap H(x_i^n, W_i^n) = \emptyset.$$

Therefore, by an application of Ramsey's Theorem, we may assume that there is an  $i$  such that (3.8) holds for all  $m \neq n$ .

Since  $H$  is a monotone normality operator, this implies that for  $m \neq n$ ,

$$(3.9) \quad H^2(x_i^n, W_i^n) \cap H^2(x_i^m, W_i^m) = \emptyset.$$

For each  $n$ , choose  $p_n \in H^4(x_i^n, W_i^N) \setminus U$ . Since  $X$  is compact, we can find a point  $p$  that is a limit point of  $\{x_n : n \in \omega\}$ .

Since the family  $\{H^2(x_i^n, W_i^n) : n < \omega\}$  is pairwise disjoint, the members of  $\{H^3(x_i^n, W_i^n) : n < \omega\}$  have pairwise disjoint closures. Thus  $p$  is a member of  $H^3(x_i^n, W_i^n)$  for at most one  $n$ .

If  $p \notin H^3(x_i^m, W_i^m)$  then  $H(p, X \setminus U)$  and  $H^4(x_i^m, W_i^m)$  are disjoint, and so  $H(p, X \setminus U)$  is an open neighborhood of  $p$  that contains at most one member of  $\{p_n : n < \omega\}$ , a contradiction.  $\square$

**Proposition 3.7.** If  $K \in X/M$  is an equivalence class of size  $\geq 3$ , then there is a co-zero set  $U \in M$  that shatters  $K$ .

*Proof.* Note that since  $X$  is separable, an equivalence class in  $X/M$  has empty interior, except in the case where the equivalence class consists of a single isolated point from  $X$ . If  $K \in X/M$  has size  $\geq 3$ , choose distinct  $\{x_i : i < 3\}$  in  $K$ , and choose open sets (not necessarily from  $M$ !)  $\{W_i : i < 3\}$  with disjoint closures such that  $x_i \in W_i$ .

Since  $K$  is the intersection of all co-zero sets from  $M$  that contain it and  $K$  has empty interior, for each  $i < 3$  we can find a cozero set  $U_i \in M$  such that  $K \subseteq U_i$  and  $H^4(x_i, W_i) \not\subseteq U_i$ . Finally, the set  $U = U_0 \cap U_1 \cap U_2$  is a cozero set in  $M$  with all the required properties.  $\square$

From the two preceding propositions, it follows that all but countably many equivalence classes in  $X/M$  are of size 2 or smaller, but we need to improve this to *all* equivalence classes. The proof of this breaks into two steps — first we use a variant of an argument from Rudin's paper [15] to show that there is at least one  $M$  for which every  $=_M$ -class has size at most 2, and then we show that in fact it must hold for every such  $M$ .

**Proposition 3.8.** There is a countable elementary submodel  $M$  of  $H(\chi)$  containing  $X$  such that all  $=_M$ -classes are of cardinality  $\leq 2$ .

*Proof.* Assume by way of contradiction that the result fails, and let  $\langle M_\alpha : \alpha < \omega_1 \rangle$  be an increasing and continuous  $\in$ -chain of countable elementary submodels of  $H(\chi)$  such that  $X \in M_0$  and  $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$ .

By our assumption, we can choose for each  $\alpha < \omega_1$  an  $=_\alpha$ -class<sup>6</sup>  $K_\alpha \in M_{\alpha+1}$  containing at least three elements. Since  $M_\alpha \in M_\beta$  for  $\alpha < \beta$ , it follows that every  $=_\beta$ -class is contained in a unique  $=_\alpha$ -class. Thus,

$$(3.10) \quad \alpha < \beta < \omega_1 \implies \text{either } K_\alpha \cap K_\beta = \emptyset \text{ or } K_\beta \subseteq K_\alpha.$$

Since  $\omega_1 \rightarrow (\omega_1, \omega)^2$  by the Dushnik–Miller Theorem<sup>7</sup> we know that either there is an infinite  $A \subseteq \omega_1$  such that  $K_\alpha \cap K_\beta = \emptyset$  for  $\alpha < \beta$  in  $A$ , or there is

---

<sup>6</sup>We write  $=_\alpha$  instead of  $=_{M_\alpha}$ .

<sup>7</sup>From [7], or see Theorem 14.6 of [12]

an uncountable set  $B \subseteq \omega_1$  such that  $K_\beta \subseteq K_\alpha$  for  $\alpha < \beta$  in  $B$ . We will get a contradiction by proving that both of these alternatives are untenable.

We will dispose of the second alternative first, so assume that we have such an uncountable  $B$ . Our argument consists of showing that  $K_\beta$  is actually a *proper* subset of  $K_\alpha$  for  $\alpha < \beta < \omega_1$ , and then quoting an old result due both to Ostaszewski and to Moody.

Suppose now that  $\alpha < \beta$  and  $K_\beta \subseteq K_\alpha$ . Since  $K_\alpha \in M_{\alpha+1}$  and  $M_{\alpha+1}$  is countable, we know that  $K_\alpha \in M_\beta$  as well, and hence

$$(3.11) \quad M_\beta \models |K_\alpha| \geq 3.$$

In particular, we can find points  $x \neq y$  in the set  $M_\beta \cap K_\alpha$ . These two points are separated by a continuous function in  $M_\beta$ , and hence  $K_\beta$  can contain at most one of them. Thus,  $K_\beta$  is a proper subset of  $K_\alpha$ .

This implies that the sequence  $\langle K_\alpha : \alpha \in B \rangle$  is an uncountable strictly decreasing sequence of closed subsets of  $X$ . However, this is absurd, as by Ostaszewski [14] and Moody [13] a separable monotonically normal space is hereditarily Lindelöf.<sup>8</sup>

The other alternative available to us is that there is an infinite  $A$  such that  $\{K_\alpha : \alpha \in A\}$  is pairwise disjoint. This case is disposed of by essentially the same argument used in the proof of Proposition 3.6, so we leave it to the reader.

Since either alternative leads to a contradiction, it must be the case that there is an  $\alpha < \omega_1$  for which all  $=_\alpha$ -classes are of cardinality at most 2 and this establishes the proposition  $\square$

Finally, to show that in fact the conclusion of Proposition 3.8 holds for *every* such model  $M$ , we note that the definition of our equivalence relations  $=_M$  doesn't depend on the fact that  $M$  is an elementary submodel of  $H(\chi)$ . In fact, we can carry out the same construction given *any* set of continuous real-valued functions defined on  $X$ . Thus, we can view Proposition 3.8 as stating that there is a countable set  $\mathcal{X}$  of continuous real-valued functions defined on  $X$  for which the associated equivalence classes all have cardinality at most 2 — simply take  $\mathcal{X}$  to be the set of all such functions in  $M$ .

Now let  $M$  be an *arbitrary* countable elementary submodel of  $H(\chi)$  containing  $X$ . By elementarity,

$$(3.12) \quad M \models \text{"there is a countable set } \mathcal{X} \text{ as above"}. .$$

Since  $\mathcal{X}$  is countable, every member of  $\mathcal{X}$  is also in  $M$ . Thus, any  $=_M$ -equivalence class is contained in a  $=_\mathcal{X}$ -equivalence class, and therefore each such  $=_M$ -class is of cardinality at most 2. Thus, the proof of Theorem 2 is complete.  $\square$

---

<sup>8</sup>In fact, they show that  $c(X) = hc(X) = hL(X)$  for  $X$  monotonically normal. Gartside's paper [9] contains an extensive treatment of cardinal invariants of monotonically normal spaces.

In closing, we remark that the rest of Rudin's proof of Nikiel's Conjecture seems to be amenable to a similar treatment, and we plan to examine this in a future paper.

#### REFERENCES

- [1] Ingo Bandlow. A note on applications of the Löwenheim-Skolem theorem in general topology. *Z. Math. Logik Grundlag. Math.*, 35(3):283–288, 1989.
- [2] Ingo Bandlow. A characterization of Corson-compact spaces. *Comment. Math. Univ. Carolin.*, 32(3):545–550, 1991.
- [3] Ingo Bandlow. A construction in set-theoretic topology by means of elementary substructures. *Z. Math. Logik Grundlag. Math.*, 37(5):467–480, 1991.
- [4] Alan Dow. An introduction to applications of elementary submodels to topology. *Topology Proc.*, 13(1):17–72, 1988.
- [5] Alan Dow. Set theory in topology. In *Recent progress in general topology (Prague, 1991)*, pages 167–197. North-Holland, Amsterdam, 1992.
- [6] Alan Dow. More set-theory for topologists. *Topology Appl.*, 64(3):243–300, 1995.
- [7] Ben Dushnik and E. W. Miller. Partially ordered sets. *Amer. J. Math.*, 63:600–610, 1941.
- [8] Ryszard Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989.
- [9] P. M. Gartside. Cardinal invariants of monotonically normal spaces. *Topology Appl.*, 77(3):303–314, 1997.
- [10] Gary Gruenhage. Generalized metric spaces. In *Handbook of set-theoretic topology*, pages 423–501. North-Holland, Amsterdam, 1984.
- [11] Gary Gruenhage. Generalized metric spaces and metrization. In *Recent progress in general topology (Prague, 1991)*, pages 239–274. North-Holland, Amsterdam, 1992.
- [12] András Hajnal and Peter Hamburger. *Set theory*, volume 48 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1999. Translated from the 1983 Hungarian original by Attila Máté.
- [13] P. J. Moody. A construction that yields a nonacyclic monotonically normal space. In *Proceedings of the Symposium on General Topology and Applications (Oxford, 1989)*, volume 44, pages 263–269, 1992.
- [14] A. J. Ostaszewski. Monotone normality and  $G_\delta$ -diagonals in the class of inductively generated spaces. In *Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978)*, volume 23 of *Colloq. Math. Soc. János Bolyai*, pages 905–930. North-Holland, Amsterdam, 1980.
- [15] Mary Ellen Rudin. Nikiel's conjecture. *Topology Appl.*, 116(3):305–331, 2001.

DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OH 45701  
*E-mail address:* eisworth@math.ohio.edu